

Problem 6

A *subtangent* is a portion of the x -axis that lies directly beneath the segment of a tangent line from the point of contact to the x -axis. Find the curves that pass through the point $(c, 1)$ and whose subtangents all have length c .

Solution

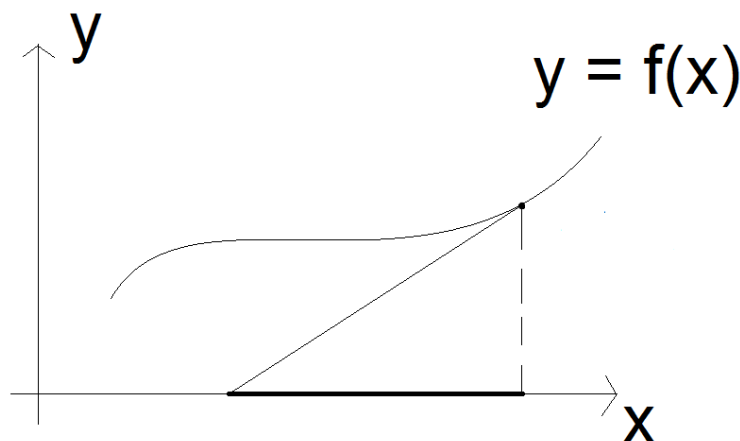


Figure 1: This is a figure illustrating a random subtangent (in bold).

We know from algebra that if we have two points, (x_1, y_1) and (x_2, y_2) , then the slope of the line going through them is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The first point is $(x - c, 0)$, the x -intercept, and the second point is (x, y) , a random point on the graph. The slope at x is dy/dx . So

$$\begin{aligned} \frac{dy}{dx} &= \frac{y - 0}{x - (x - c)} \\ \frac{dy}{dx} &= \frac{y}{c}. \end{aligned}$$

Solve with separation of variables.

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{c} \\ \ln |y| &= \frac{x}{c} + D \\ |y| &= e^{x/c} e^D \\ y(x) &= Ae^{x/c} \end{aligned}$$

Since the graph has to go through $(c, 1)$, this means that $y(c) = 1$, and the constant of integration can be determined. $y(c) = Ae = 1$, which means $A = e^{-1}$. Therefore,

$$y(x) = e^{\frac{x}{c} - 1}.$$